Midterm Exam II - Review

MATH 125 - Spring 2022

Midterm 2 covers Chapter 3, Sections 4.1-4.4 and 4.6

Midterm Exam 1: Tuesday 4/12, 5:50-7:50 pm in Wescoe 3140 and Strong 330

The following is a list of important concepts that will be tested on Midterm Exam 2. This is not a complete list of the material that you should know for the course, but it is a good indication of what will be emphasized on the exam. A thorough understanding of all of the following concepts will help you perform well on the exam. Some places to find problems on these topics are the following: in the book, in the slides, in the homework, on quizzes, and Achieve.

- **The Basics of Derivatives: Sections 3.1-3.8**

  Differentiation rules for general functions $f(x)$ and $g(x)$.

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x) \quad \text{(product rule)}$$

$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} \quad \text{(quotient rule)}$$

$$\frac{d}{dx}(c\,f(x)) = cf'(x)$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x) \quad \text{(chain rule)}$$

  Derivatives of particular functions.

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln(x)) = \tfrac{1}{x}$$

$$\frac{d}{dx}(x^n) = n\,x^{n-1} \qquad \frac{d}{dx}(a^x) = a^x\,\ln(a) \qquad \frac{d}{dx}(\log_a(x)) = \tfrac{1}{x\ln(a)}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x) \qquad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

Much of the content on this exam requires you to differentiate a variety of functions using the rules in the boxes above. This may include a combination of many of the above differentiation rules to differentiate a single function. For example, you may need to use the product rule, then the chain rule twice, and finally a quotient rule when differentiating $f(x) = \sin^2(x)\tan\left(\frac{\ln(x)}{x}\right)$.

You still need to know the interpretations of the derivative of a function as the slope of a tangent line and the instantaneous rate of change. You should be comfortable working with these differentiation rules for functions that are not explicitly defined. For example, you should be able to use the chain rule to compute the derivative of $F(x) = f(\sqrt{x})$ as $F'(x) = \frac{f'(\sqrt{x})}{2\sqrt{x}}$.
Implicit Differentiation is an application of the Chain Rule. Given an implicit equation involving x and y, you should be able to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in term of x and y. You should be able to use $\frac{dy}{dx}$ to find the equation of the line tangent to the curve at a point (x_0, y_0) defined by the implicit equation. Implicit differentiation was used to find the derivative of the inverse trigonometric functions and $\log_a(x)$.

Whenever the variable being differentiated differs from the variable that we are differentiating with respect to, a new derivative term is produced. For example, $\dfrac{d}{dz}(r^3) = 3r^2\dfrac{dr}{dz}$.

1. Find $\dfrac{dy}{dx}$, $\dfrac{dx}{dy}$, and $\dfrac{dx}{dt}$ for the equation

   (a) $xy + x^2y^2 = 6$

$\dfrac{dy}{dx}$: $y + x\dfrac{dy}{dx} + 2xy^2 + 2x^2y\dfrac{dy}{dx} = 0$

Regroup:

$y + 2xy^2 = -(x + 2x^2y)\dfrac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = -\frac{y + 2xy^2}{x + 2x^2y}}$$

$\dfrac{dx}{dy}$: $\boxed{\dfrac{dx}{dy} = -\dfrac{x + 2x^2y}{y + 2xy^2}}$

$\dfrac{dx}{dt}$: $y\dfrac{dx}{dt} + x\dfrac{dy}{dt} + 2xy^2\dfrac{dx}{dt} + 2x^2y\dfrac{dy}{dt} = 0$

$$\frac{dx}{dt} = -\frac{x\dfrac{dy}{dt} + 2x^2y\dfrac{dy}{dt}}{y + 2xy^2}$$

   (b) $e^{xy} = \sin(y^2)$

$\dfrac{dy}{dx}$: $e^{xy}(y + x\dfrac{dy}{dx}) = 2\cos(y^2)y\dfrac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} - 2y\cos(y^2)}}$$

$\dfrac{dx}{dy}$: $\boxed{\dfrac{dx}{dy} = -\dfrac{xe^{xy} - 2y\cos(y^2)}{ye^{xy}}}$

$\dfrac{dx}{dt}$: $e^{xy}\left(\dfrac{dx}{dt}y + x\dfrac{dy}{dt}\right) = 2y\cos(y^2)\dfrac{dy}{dt}$

$$\frac{dx}{dt} = \frac{-xe^{xy}\dfrac{dy}{dt} + 2y\cos(y^2)\dfrac{dy}{dt}}{ye^{xy}}$$

2. Find the equation of the line tangent to the curve $\sqrt{x} + \sqrt{y} = 5$ at $(9, 4)$.

$$\frac{d}{dx}: \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx}\right|_{(9,4)}: \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\left.\frac{dy}{dx}\right|_{(9,4)} = 0$$

Plug in: $\frac{1}{6} + \frac{1}{4}\frac{dy}{dx} = 0$

Slope: $\left.\frac{dy}{dx}\right|_{(9,4)} = -\frac{2}{3}$

Tangent: $y - 4 = -\frac{2}{3}(x - 9)$

$$\boxed{y = -\frac{2}{3}x + 10}$$

3. Find $\frac{d^2y}{dx^2}$ for the curve $\sqrt{x} + \sqrt{y} = 5$.

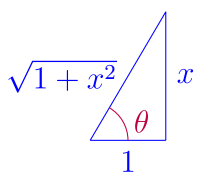
$$\frac{d}{dx}: \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

Solve: $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = -x^{-1/2}y^{1/2}$

$$\frac{d^2}{dx^2}: \quad \begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{2}\left(y^{-1/2}x^{-1/2}\frac{dy}{dx} - y^{1/2}x^{-3/2}\right) \\ &= -\frac{1}{2}\left(y^{-1/2}x^{-1/2}(-x^{-1/2}y^{1/2}) - y^{1/2}x^{-3/2}\right) \\ &= \frac{1}{2}(y^{1/2}x^{-3/2} + x^{-1}) \\ &= \frac{\sqrt{y} + \sqrt{x}}{2x\sqrt{x}} \end{aligned}$$

4. Simplify $\sec(\arctan(x))$ and then find $\frac{dy}{dx}$ of $y = \arctan(x)$ using implicit differentiation.

$\sec(\arctan(x))$:



$$\begin{aligned} \theta &= \arctan(x) \\ &\text{so} \\ \sec(\arctan(x)) &= \sec(\theta) \\ &= \frac{1}{\cos(\theta)} \\ &= \frac{1}{\frac{1}{\sqrt{1+x^2}}} \\ &= \sqrt{1+x^2} \end{aligned}$$

Derivative $\tan(y) = x$

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\sec^2(\arctan(x))\frac{dy}{dx} = 1$$

$$(\sqrt{1+x^2})^2\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

5. Find the derivative with respect to x of the following functions:

$y = e^{x\sin(x)}$	$y = \frac{\sin(\pi x)}{x}$	$y = \frac{3x-2}{\sqrt{2x+1}}$
$y = \tan^2(\sin(x))$	$y = \arctan(x^2 + \sqrt{x})$	$y = 3^{x\ln(x)}$

$$y = e^{x\sin(x)}:\quad y' = e^{x\sin(x)}(\sin(x) + x\cos(x))$$

$$y = \frac{\sin(\pi x)}{x}:\quad y' = \frac{\pi x\cos(\pi x) - \sin(\pi x)}{x^2}$$

$$y = \frac{3x-2}{\sqrt{2x+1}}\quad y' = \frac{\sqrt{2x+1}(3) - \dfrac{2(3x-2)}{2\sqrt{2x+1}}}{2x+1}$$

$$y = \tan^2(\sin(x)):\quad y' = 2\tan(\sin(x))\sec^2(\sin(x))(\cos(x))$$

$$y = \arctan(x^2+\sqrt{x}):\quad y' = \frac{2x + \dfrac{1}{2\sqrt{x}}}{(x^2+\sqrt{x})^2+1}$$

$$y = 3^{x\ln(x)}:\quad y' = 3^{x\ln(x)}\ln(3)\,(\ln(x)+1)$$

6. Find a point on $x^3 + y^3 = 3xy$ other than the origin at which the tangent line is horizontal.

$\frac{d}{dx}$: $3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$

$\frac{dy}{dx}$:

Isolate: $3x^2 - 3y = 3x\frac{dy}{dx} - 3y^2\frac{dy}{dx}$

Solve: $\frac{dy}{dx} = -\frac{3x^2-3y}{3y^2-3x}$

$\frac{dy}{dx} = 0$ $3x^2 - 3y = 0$ so $y = x^2$.

Plug in: $x^3 + y^3 = 3xy$ implies $x^3 + x^6 = 3x^3$

Combine: $x^6 - 2x^3 = 0$ implies $x^3(x^3-2) = 0$

Answer: $x = \sqrt[3]{2}$ and $y = x^2 = \sqrt[3]{4}$

At $(\sqrt[3]{2}, \sqrt[3]{4})$, the graph has a horizontal tangent line.

• Logarithmic Differentiation: Section 3.9

When differentiating an expression with variables in both the base and the exponent, take the natural logarithm of both sides, use the properties of logarithms to simplify, and then implicitly differentiate.

$a^{\log_a(x)} = x$ and $\log_a(a^x) = x$,

$\log_a(xy) = \log_a(x) + \log_a(y)$,

$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$,

$\log_a(x^y) = y\log_a(x)$.

Exercies:

1. Differentiate the following using logarithmic differentiation

(a) $y = x\log_2(x)$

Product rule & log Rule:

$$\frac{dy}{dx} = \log_2(x) + x\frac{1}{x\ln(2)}$$

$$y' = \boxed{\log_2(x) + \frac{1}{\ln(2)}}$$

(b) $y = \sqrt{\frac{x(2x+3)^5}{(7x-10)^{15}}}$.

Logarithmic Differentiation:

Take the log of both sides:

$$\ln(y) = \ln\left(\sqrt{\frac{x(2x+3)^5}{(7x+10)^{15}}}\right)$$

Simplify using the logarithmic rules:

The log of power rule: $\ln(y) = \frac{1}{2}\ln\left(\frac{x(2x+3)^5}{(7x+10)^{15}}\right)$

The log of quotient rule:

$$= \frac{1}{2}\left(\ln\left(x(2x+3)^5\right) - \ln\left((7x+10)^{15}\right)\right)$$

The log of product rule:

$$= \frac{1}{2}\left(\ln(x) + \ln\left((2x+3)^5\right) - \ln\left((7x+10)^{15}\right)\right)$$

$$= \frac{1}{2}\left(\ln(x) + 5\ln\left((2x+3)\right) - 15\ln\left((7x+10)\right)\right)$$

Derivative of the simplified form:

$$\frac{\frac{dy}{dx}}{y} = \frac{1}{2}\left(\frac{1}{x} + 5\frac{2}{2x+3} - 15\frac{7}{7x+10}\right)$$

Multiply both sides by y

$$\frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x} + 5\frac{2}{2x+3} - 15\frac{7}{7x+10}\right)$$

$$\frac{dy}{dx} = \frac{\sqrt{\frac{x(2x+3)^5}{(7x-10)^{15}}}}{2}\left(\frac{1}{x} + 5\frac{2}{2x+3} - 15\frac{7}{7x+10}\right) \quad \leftarrow \text{Replace } y = \sqrt{\frac{x(2x+3)^5}{(7x-10)^{15}}}$$

or by algebra: $\boxed{\frac{dy}{dx} = \sqrt{\frac{x(2x+3)^5}{4(7x-10)^{15}}}\left(\frac{1}{x} + 5\frac{2}{2x+3} - 15\frac{7}{7x+10}\right)}$

(c) $y = x^{x^2}$

Logarithmic Differentiation

Log: $\ln(y) = x^2\ln(x)$

$$\frac{d}{dx}: \frac{y'}{y} = \underbrace{2x\ln(x) + x^2\frac{1}{x}}_{\text{Product Rule}}$$

Solve: $y' = \boxed{(2x\ln(x) + x)x^{x^2}}$

(d) $y = \cos(x)^{\sqrt{x}}$

Logarithmic Differentiation

Take the natural log of both sides:

$\ln(f(x) = \sqrt{x}\ln(\cos(x))$

Take the derivative of both sides:

$$\frac{f'(x)}{f(x)} = \frac{1}{2\sqrt{x}}\ln(\cos(x)) - \sqrt{x}\frac{\sin(x)}{\cos(x)}$$

Multiply both sides by $y = f(x)$:

$$f'(x) = \boxed{\frac{\cos(x)^{\sqrt{x}}}{2\sqrt{x}}\ln(\cos(x)) - \sqrt{x}\frac{\cos(x)^{\sqrt{x}}}{x}\tan(x)}$$

(e) $y^x = x^y$

Logarithmic Differentiation:

Log: $x\ln(y) = y\ln(x)$

$$\frac{d}{dx}: \underbrace{\ln(y) + \frac{xy'}{y}}_{\text{Product Rule}} = \underbrace{y'\ln(x) + \frac{y}{x}}_{\text{Product Rule}}$$

Solve: $y' = \dfrac{\frac{y}{x} - \ln(y)}{\frac{x}{y} - \ln(x)}$

- ## Related Rates: Section 3.10

Solving a related rates problem:

Step 1) Draw a diagram of the problem and define the variables.

Step 2) Find an equation relating the variables from Step 1.

Step 3) Differentiate both sides of the equation from Step 2 with respect to t.

Step 4) Determine the known quantities given in the problem, plug them into the equation from Step 3, and solve for the unknown quantity.

Note that in Steps 1-3 the variables of the problem are changing as time progresses, but in Step 4 we plug in particular values for the variables, so they are fixed.

Formulas and Concepts:

- Areas and Volumes:

$$\text{Area Rectangle} = \text{Length} \times \text{Width} \qquad \text{Area Circle} = \pi\times \text{Radius}^2$$

$$\text{Surface Area Sphere} = 4\pi\times \text{Radius}^2 \qquad \text{Volume Sphere} = \tfrac{4\pi}{3}\times \text{Radius}^3$$

$$\text{Volume Cylinder} = \pi\times \text{Radius}^2 \times \text{Height} \qquad \text{Volume Cone} = \tfrac{\pi}{3}\times \text{Radius}^2 \times \text{Height}$$

- Perimeters, Similar Triangles, Pythagorean's Theorem (Distance Formula)
- Basic Trigonometric Functions and Identities.

Consider the following example: A snowball (which is spherical) melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Step 1) The variables in this problem are defined

D - the diameter (in cm) of the snowball at time t.
S - the surface area (in cm^2) of the snowball at time t.
$\frac{dD}{dt}$ - change in diameter per change in time (in cm/min) at time t.
$\frac{dS}{dt}$ - change in surface area per change in time (in cm^2/min) at time t.

Step 2) The equation for this problem is

$$S = \pi D^2.$$

Note that we can get this from the surface area in terms of the radius by realizing that $D/2$ is the radius, $S = 4\pi(D/2)^2$.

Step 3) Differentiating with respect to time t, we get

$$\frac{dS}{dt} = 2\pi D\frac{dD}{dt}.$$

Step 4) Now looking at the problem, the given values are

$$D = 10, \qquad S = 100\pi, \qquad \frac{dD}{dt} = ??, \qquad \text{and} \qquad \frac{dS}{dt} = -1.$$

The value of $\frac{dD}{dt}$ is what we are looking for, and we can use the formula from Step 3 to find it: plugging the values in, we get

$$-1 = 2\pi(10)\frac{dD}{dt} \quad \text{and hence} \quad \frac{dD}{dt} = -\frac{1}{20\pi}.$$

The diameter is decreasing at a rate of $\frac{1}{20\pi}$ cm/min.

Exercises:

1. A water trough has an inverted isosceles triangle as a base. This isosceles triangle has a base of 4 feet and a height of 3 feet. The trough is 7 feet long. Water is being siphoned out of the trough at a rate of 5 $\frac{ft^3}{min}$. At any time t, let H be the depth and V be the volume of water in the trough.

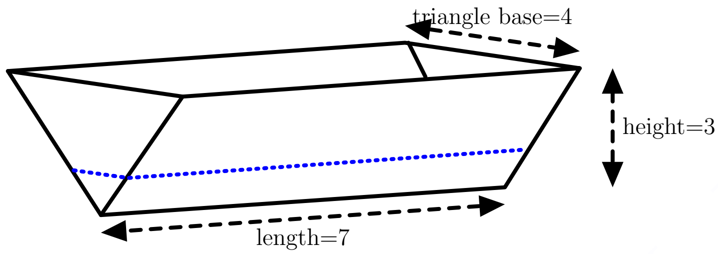
   (a) Find a formula for the length and width of the surface of the water as a function of depth of the water, H. (Hint: One of the two is constant.)

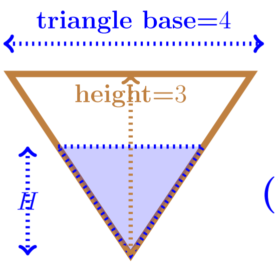
(b) Find a formula for V as a function of H.

(c) What is the depth of the water when the trough is a quarter full by volume?

(d) What is the rate of change in H at the instant when the trough is a quarter full by volume?

(e) What is the rate of change in the area of the surface of the water at the instant when the trough is $\frac{1}{4}$ full by volume?





(a) One side of rectangle is constant $\boxed{L=7}$. Let the other side be B.

By Similarity of two triangles: $\frac{H}{3} = \frac{B}{4}$ so $\boxed{B = \frac{4}{3}H}$

(b)

Note: The area of the triangle is:

$$A = \frac{1}{2}(B)(H) = \left(\frac{1}{2}\right)\left(\frac{4}{3}H\right)H = \frac{2}{3}H^2$$

$$V = A \times L = A \times 7 = \frac{2}{3}(H^2) \times 7 = \boxed{\frac{14}{3}H^2}$$

(c) The total volume of the trough$=\frac{4 \times 3 \times 7}{2}$. Now $\frac{1}{4}$ of that volume is 10.5. Set $\frac{14}{3}H^2 = 10.5$

$$H = \sqrt{\frac{10.5 \times 3}{14}} = \boxed{1.5}$$

(d) **Goal:** to find $\frac{dH}{dt}$ **Given** $\frac{dV}{dt} = -5\frac{ft^3}{min}$

Implicit Differentiation: $\frac{dV}{dt} = \frac{14}{3}(2H)\frac{dH}{dt}$

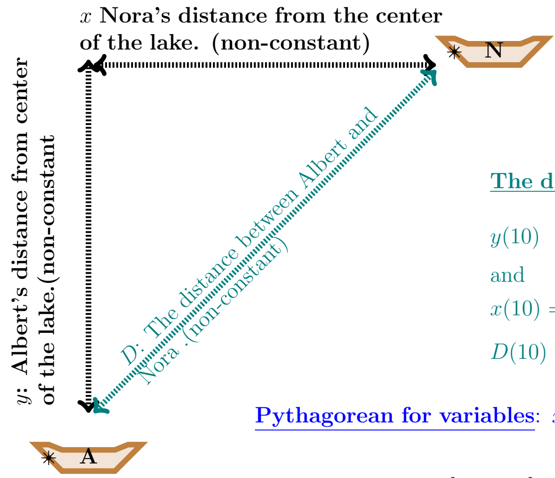
$$\frac{dH}{dt} = \frac{3 \times (-5)}{28H}$$

$$\frac{dH}{dt} = \boxed{-0.35713}\frac{ft}{min}$$

(e) $S = B \times L = \frac{28}{3}H$

$$\frac{ds}{dt} = \frac{28}{3}\frac{dH}{dt} = \frac{28}{3}(-0.35713) = \boxed{-\frac{10}{3}}\frac{ft^2}{min}$$

2. Albert and Nora are in motorboats on a lake. Albert is 5 km west of Nora. At time $t = 0$, Albert begins traveling south at a speed of 40 km/h. Five minutes later, Nora begins traveling east at a speed of 20 km/h. At what rate is the distance between them changing at $t = 10$ minutes?



Given: $\frac{dy}{dt} = 40\frac{km}{hr}$ and $\frac{dx}{dt} = 20\frac{km}{hr}$.

Want to find $\frac{dD}{dt}$

The distances after 10 minutes:

$$y(10) = \frac{40\ km/hr}{60\ min/hr} \times 10min = \frac{20}{3} \approx 6.66666\ km$$

and

$$x(10) = 5 + \frac{20\ km/hr}{60\ min/hr} \times (10-5)\ min = \frac{20}{3} \approx 6.66666\ km$$

$$D(10) = \sqrt{x(10)^2 + y(10)^2} \approx 9.42807\ km$$

Pythagorean for variables: $x^2 + y^2 = D^2$.

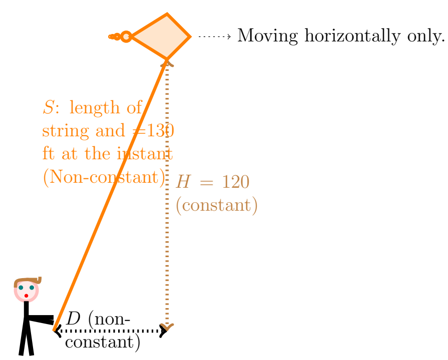
Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}$.

Plug in: $2(6.66666)(20) + 2(6.66666)(40) = 2(9.42807)\frac{dD}{dt}$

Solve: $\frac{dD}{dt} = \boxed{42.42644}$

https://youtu.be/4C4d8S9drt8 and https://youtu.be/Ce9F2_a3Svs

3. Albert is flying a kite on a string. The kite is 120 ft above Albert's hands' level and the wind is blowing the kite horizontally away from him at 6 $\frac{ft}{s}$. At what rate must he let out the string when 130 ft of string has been let out?



Let S be the length of the string and D be the horizontal distance of Albert and Kite.

Goal: Finding $\frac{ds}{dt}$ **Given:** rate=$6\frac{ft}{s}$, $s = 130ft$ and $H = 120ft$.

Variables: S and D. **Constants**: $H = 120ft$.

Pythagorean: $D^2 + 120^2 = S^2$

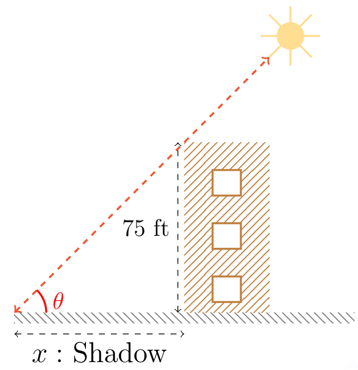
Differentiate: $2D\frac{dD}{dt} = 2S\frac{dS}{dt}$

Plug in: $2(50)(6) = 2(130)\frac{dS}{dt}$

Solve: $\frac{dS}{dt} = \frac{(50)(6)}{130} \approx 23 \times 10^{-1}$

Use Pythagorean to solve for D at the instant; $120^2 + D^2 = 130^2$ gives $D = 50$.

4. The angle of elevation of the sun is decreasing at a rate of 0.25 radians per an hour. How fast is the shadow cast by the 75 foot tall Snow Hall increasing when the angle of elevation of the sun is $\frac{\pi}{6}$?

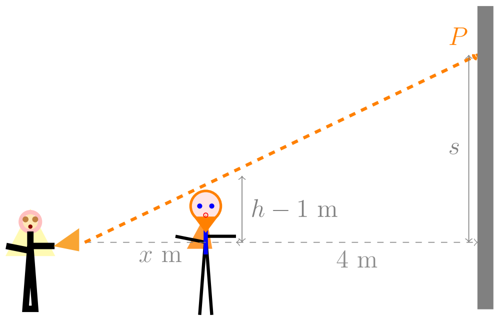


$\tan(\theta) = \dfrac{75}{x}$ and when $\theta = \frac{\pi}{6}$, $x = 75\sqrt{3}$

$$\underbrace{\sec^2(\theta)}_{\frac{1}{\cos^2(\frac{\pi}{6})}=\frac{4}{3}} \overbrace{\frac{d\theta}{dt}}^{0.25} = \underbrace{\left(\frac{-75}{x^2}\right)}_{\frac{-1}{(75)(3)}} \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = -75} \text{ ft/hr}$$

5. A woman holding a light 1 meter above the ground is shining that light at a man who is 2 meters tall. The woman is moving at $0.8\,m/s$ approaches the man from behind. The man is facing a wall which is 4 meters from in front of him. How fast is the tip P of the man's shadow moving when the woman is 3 meters from the man?

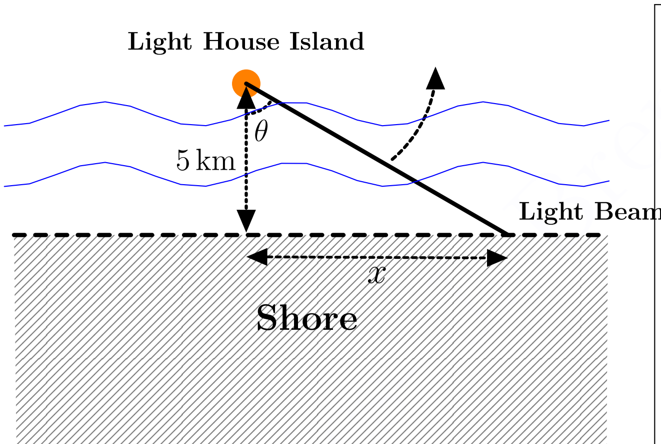


https://youtu.be/YEpk-F5yk6Q

$$\frac{x+4}{s} = \frac{x}{h-1} \text{ so } s = \frac{(x+4)(h-1)}{x}$$

$$\frac{ds}{dt} = \underbrace{\frac{-4}{x^2}}_{9}\underbrace{(h-1)}_{2-1\,m}\underbrace{\frac{dx}{dt}}_{-0.8\,m/s} \approx 0.356 \text{ m/s}$$

6. A lighthouse sits on a small island near a rocky shoreline, emitting a rotating beam of light. The lighthouse is $5\,km$ from the shore, and it emits a beam of light that rotates 12 times per minute. How quickly is the end of the light beam moving along the shoreline when $\theta = 60°$?



Given: At the instant $\theta = 60° = \frac{\pi}{3}$ rad. **Want to Find:** $\frac{dx}{dt}$

$$\frac{d\theta}{dt} = 12\,\frac{\text{rotations}}{\text{min}} = 12\,\frac{\text{rotations}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rotation}} = 24\pi\,\frac{\text{rad}}{\text{min}}$$

Using Trigonometry: $\tan(\theta) = \frac{x}{5} \longrightarrow \sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{dx}{dt}\right)$

$$\frac{dx}{dt} = 5 \times \sec^2(\theta) \times \left(\frac{d\theta}{dt}\right) = \boxed{480\pi\,\frac{\text{km}}{\text{min}}}$$

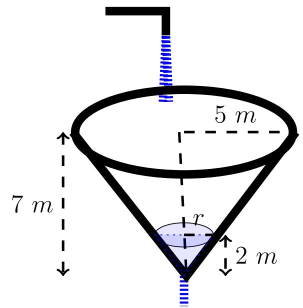
Note:

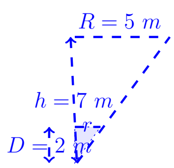
$$\sec^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{\cos(\pi/3)}\right)^2 = 4.0$$

7. Water is leaking out of a tank shaped like an inverted cone at a rate of 3 $\frac{m^3}{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 7 meters and the diameter at the top is 10 meters. If the water level is **falling** at a rate of 0.2 $\frac{m}{min}$ when the height of the water is 2 meters,

(a) use the volume formula $V = \dfrac{\pi r^2 h}{3}$ to find the rate of change in the volume of water in the cone. (Round to 3 decimal places.)

(b) find the rate at which water is being pumped into the tank.





Given: Radius of the cone: $R = 5m$ (Constant), Height of the cone: $h = 7m$(constant) and depth of the water $d = 2m$ (Non-constant)

$\frac{dD}{dt} = 0.2\frac{m}{min}$ and the rate of change in volume because of the leak$=-3\frac{m^3}{min}$

By similarity of triangles :

Solve for r: $\frac{D}{h} = \frac{r}{R} \implies r = \frac{D \times R}{h} \implies r = \frac{5D}{7}$

At the instant where $D = 2$, $\implies r = \frac{10}{7}$

(a)

Relate the volume to depth: $V(D) = \frac{\pi r^2 D}{3}$

Substitute r by a function of D: $V(D) = \frac{(\frac{5D}{7})^2 D}{3} \implies V(D) = \frac{25\pi D^3}{147}$

Differentiate: $\frac{dV}{dt} = \frac{25\pi D^2}{49}\left(\frac{dD}{dt}\right)$

Substitute the values $\frac{dV}{dt} = \frac{25\pi(2^2)}{49}(-0.2) = \boxed{-\frac{20\pi}{49}} \approx -1.28\ \frac{m^3}{min}$

(b)

$\frac{dV}{dt} = rate_{in} - rate_{out}$

$-\frac{20\pi}{49} = rate_{in} - 3 \implies \boxed{rate_{in} = 3 - \frac{20\pi}{49}} \approx 1.72\ \frac{m^3}{min}$

Note that for this problem it is important to solve r as a function of D and then differentiate.

- **Approximating with Tangent Lines: Section 4.1**

  You should know how to work with linear approximations and differentials.

  – Given a function $f(x)$, the linear approximation for $f(x)$ at $x = a$ is the line tangent to $f(x)$ at a. That is, the linear approximation of $f(x)$ at $x = a$ is

$$L_a(x) = f(a) + f'(a)(x - a)$$

  This is the line that best approximates the function $f(x)$ when x is "near" a.

  – Given a function $y = f(x)$, the differential dy is defined $dy = f'(x)dx$. Although it is not technically correct, you can think of this as "multiplying both sides" of $\frac{dy}{dx} = f'(x)$ by dx. In this situation dx represents a small change in x and dy represents the approximate resulting small change in y.

    * Be able to use differentials to approximate the values of certain functions.
    * Be able to use differentials to absolute and relative error.

1. Approximate the value of $\sqrt{10001}$.

The function to use $f(x) = \sqrt{x}$ and we use the tangent line at point $a = 10000$ and $\Delta x = 1$:

$$y = f(a) + f'(a)\Delta x \implies L(10000)(x) = f(10000) + f'(10000)(10001 - 10000) \underset{f'(x)=\frac{1}{2\sqrt{x}}}{\implies}$$

$$y = \sqrt{10000} + \frac{1}{2\sqrt{10000}}(1) = 100 + \frac{1}{2(100)}(1)$$

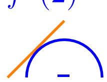
$$f(10001) \approx \boxed{100.005}$$

2. Estimate $f(2.1)$ if $f(2) = 1$ and $f'(2) = 3$; is it an over or under estimate if $f''(2) = -2$?

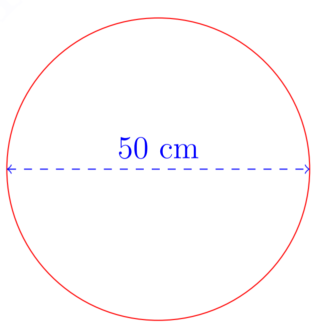
$f(2.1) \approx L_{(2)}(x) = f(2) + f'(2)(x - 2)$

$f(2.1) \approx L_{(2)}(2.1) = f(2) + f'(2)(2.1 - 2) = 1 + 3(0.1) = 1.3$

$f''(2) = -2 < 0$ ☹ so the tangent line is above the graph and tangent line is over-estimating.



3. Five Star Pizza claims that their pizzas are circular with a diameter of 50cm. Estimate the quantity of pizza lost or gained if the diameter is off by at most 1.2cm.

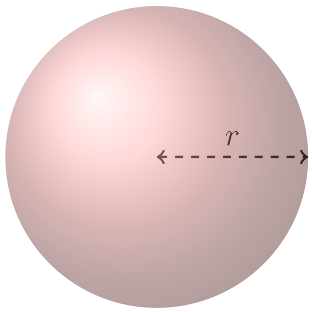


– $d = 50$ $\Delta d = \pm 1.2$

– $A(d) = \frac{\pi d^2}{4}$ and $A'(d) = \frac{\pi d}{2}$

– $\Delta A \approx dA = \underbrace{A'(50)}_{25\pi}\underbrace{\Delta d}_{\pm 1.2} = \pm 30\pi \ cm^2$

4. The radius of a spherical ball is measured at $r = 15$cm. If the ball is to be painted, covering the ball in 0.3cm of paint, use differentials to estimate the amount of paint used on the ball.



$$V = \frac{4\pi r^3}{3} \underset{\text{Derivative}}{\Longrightarrow}$$
$$\frac{dV}{dt} = 4\pi r^2$$
$$\Delta V \approx dV = \underbrace{\pi(4)(15)^2}_{V(15)}\underbrace{(0.3)}_{\Delta r} = \boxed{270\pi}\ unit^3$$

5. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the surface area of the sphere?

$r = 21$ and $dr = \pm 0.05$

$SA(r) = 4\pi r^2$ so $SA'(r) = 8\pi r$

$dSA = \underbrace{8\pi(21)}_{SA'(21)}\underbrace{(\pm 0.05)}_{\Delta r} = 8.4\ cm^2$

6. The dosage D of diphenhydramine for a dog of body mass w kg is $D = 4.7w^{\frac{2}{3}}$ mg. Estimate the maximum allowable error in w for a dog of mass $w = 10$kg if the percentage error in D must be less than 3%.

- $D(w) = 4.7w^{2/3}$ mg
- $D'(w) = 3.13w^{-1/3}$
- $dD = D'dw$
- $D(10) = 21.82$ mg
- $D'(10) = 1.45$ mg
- $\pm 0.03 = \text{Relative ErrorRelative Error}\dfrac{dD}{D(10)} = \dfrac{D'(10)\Delta w}{D(10)}$
- $\Delta w = \dfrac{\pm 0.03 D(10)}{D'(10)} = \pm 0.45$ kg

• The Shape of a Curve: Sections 4.2-4.4

Be familiar with the definition of the following: Increasing, decreasing, absolute maximum/minimum value, local maximum/minimum value, and critical number. Be sure that you are aware of the difference between a maximum/minimum <u>value</u> and the point at which the maximum/minimum value occurs; the first derivative test is the most common method of verifying local extrema.

(Fermat) If $f(c)$ is a local extrema and f is differentiable, then $f'(c) = 0$.

- ∗ Local extrema occur at critical points.
- ∗ Not all critical points are at local extrema, but all local extrema occur at critical points.
- ∗ Finding critical points is how we locate local extrema.

(Extreme Value) If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

- ∗ You can find the absolute extrema by following the **Closed Interval Method**.

(Mean Value) If f is differentiable on (a, b) and continuous on $[a, b]$, then there exists c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

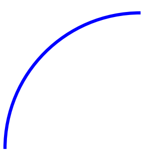
- ∗ For a differentiable function on an interval, there always exists a point whose instantaneous rate of change equals the average rate of change.

(2nd Derivative) Suppose f'' is continuous near c and $f'(c) = 0$. If $f''(c) > 0$ then c is a local minimum of f and if $f''(c) < 0$ then c is a local maximum of f.

Given a function f, be able to find intervals where f falls into one of the following categories and identify the shape of the graph there:

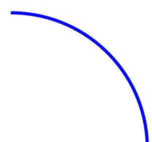
f is increasing and concave down

$f'(x) > 0$ and $f''(x) < 0$



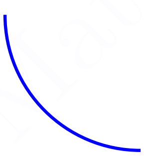
f is decreasing and concave down

$f'(x) < 0$ and $f''(x) < 0$



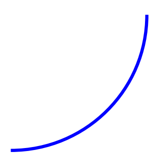
f is decreasing and concave up

$f'(x) < 0$ and $f''(x) > 0$



f is increasing and concave up

$f'(x) > 0$ and $f''(x) > 0$



Exercises:

1. Suppose $f(2) = 1$ and $f'(x) \geq 3$ for all x. What is the smallest that $f(6)$ can be?

Write the statement of MVT:

f is differentiable on $[2, 6]$ so the re exists an x-value c such that $\dfrac{f(6) - f(2)}{6 - 2} = f'(c)$

Use the bounds for f':

$$3 \leq f'(x) \implies 3 \leq f'(c) \implies 3 \leq \frac{f(6) - f(2)}{6 - 2}$$

Replace the value for $f(2)$: $3 \leq \dfrac{f(6) - 1}{4} \implies 3 \leq \dfrac{f(6) - 1}{4}$

Solve the inequality:

$$3 \leq \frac{f(6) - 1}{4} \implies (4)(3) + 1 \leq f(6) \implies \boxed{13 \leq f(6)}$$

2. Prove that $c = 4$ is the largest real root of $f(x) = x^4 - 8x^2 - 128$.

$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$ and $x = 0, -2, 2$ are the critical points.

By the way of contradiction, let $m > 4$ be a larger zero. Then by MVT, $\dfrac{f(m) - f(4)}{m - 4} = f'(c)$ for $4 < c < m$. But $f(m) - f(4) = 0 - 0 = 0$ so $f'(c) = 0$ but that is not possibe.

3. Find the extreme values of $f(x) = 2x^3 - 9x^2 + 12x$ on $[0, 2]$ and then on $[0, 3]$.

$f'(x) = 6x^2 - 18x + 12 = 6(x-2)(x-1)$ so c.p: $x = 1, 2$

Points	x	$f(x)$	Classification
End Point	0	$f(0) = 0$	Absolute Min
End Point	2	$f(2) = 4$	
Critical Point	1	$f(1) = 5$	Absolute Max

Points	x	$f(x)$	Classification
End Point	0	$f(0) = 0$	Absolute Min
End Point	3	$f(3) = 9$	Absolute Max
Critical Point	1	$f(1) = 5$	

4. Find the critical points of $f(x) = \sin(x) + \cos(x)$ and determine the extreme values of f on $[0, \frac{3\pi}{4}]$.

C.P: $f'(x) = \cos(x) - \sin(x)$ and $\cos(x) = \sin(x)$ implies $\tan(x) = 1$. General solution is $x = \dfrac{\pi}{4} + k\pi$ but only $\dfrac{\pi}{4}$ is the only one in the domain.

IVT:

Points	x	$f(x)$	Classification
End Point	0	1	
End Point	$\frac{3\pi}{4}$	0	Absolute Min
Critical Point	$\frac{\pi}{4}$	$\sqrt{2}$	Absolute Max

5. Suppose f'' is continuous everywhere.

(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?

$f''(2) = -5$ ☹ at a critical point, it means local max.

(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

$f''(6) = 0$ means the second derivative is inconclusive.

6. Let $f(x) = x^{\frac{1}{3}}(8 + x)$.

(a) Verify that $f'(x) = \frac{4}{3}\left(\frac{x+2}{x^{\frac{2}{3}}}\right)$

$\underset{\text{Distribute the original}}{\Longrightarrow}$ $f(x) = 8x^{\frac{1}{3}} + x^{\frac{4}{3}}$ $\underset{\text{Take the derivative:}}{\Longrightarrow}$ $f'(x) = \frac{8}{3}x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}}$ $\underset{\text{Factor:}}{\Longrightarrow}$ $f'(x) = \frac{4}{3}x^{\frac{-2}{3}}(2 + x)$

$\underset{\text{Rewrite the negative exponent:}}{\Longrightarrow}$ $f'(x) = \frac{4}{3}\left(\frac{x+2}{x^{\frac{2}{3}}}\right)$

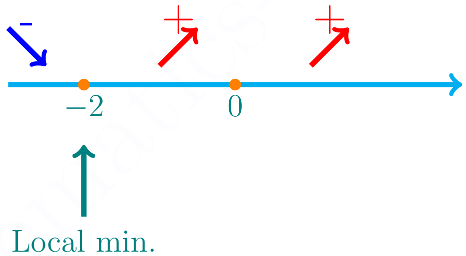
(b) Identify and classify the local extrema of $f(x)$.

Find the critical values:

$f'(x) = 0 \implies \boxed{x = -2}$

f' does not exists $\implies \boxed{x = 0}$

Use the first derivative test :



Plug in the original to find y-value: $f(-2) = 8(-2)^{1/3} + (-2)^{4/3} = -6\sqrt[3]{2}$

Local Min: $\boxed{(-2, -6\sqrt[3]{2})}$

(c) Find the absolute extrema of $f(x)$ on the interval $[-27, 0]$.

Check the Extreme Value Theorem conditions: f is continuous everywhere and the interval is closed so we can use the theorem.

List and check all critical values within the interval and the end points.

$\underbrace{}_{}^{0}$, $\underbrace{}^{-2}$, $\underbrace{}^{-27}$

$f(0) = \boxed{0}$ Neither, $f(-2) = \boxed{-6\sqrt[3]{2}}$ Smallest, $f(-27) = \boxed{57}$ Largest

Find the absolute Extrema

The absolute minimum point is $\boxed{(-2, -6\sqrt[3]{2})}$.

The absolute Maximum point is $\boxed{(-27, 57)}$.

Video of a Similar Problem: https://mediahub.ku.edu/media/t/0_e8semsg1

7. The Shape of a Curve: Sections 4.2-4.4, 4.6

Be familiar with the definition of the following: Increasing, decreasing, absolute maximum/minimum value, local maximum/minimum value, and critical number. Be sure that you are aware of the difference between a maximum/minimum value and the point at which the maximum/minimum value occurs; the first derivative test is the most common method of verifying local extrema.

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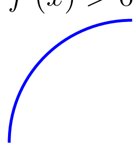
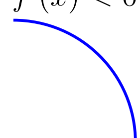
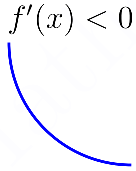


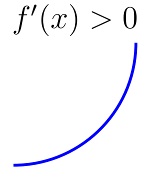
f is decreasing and concave down
$f'(x) < 0$ and $f''(x) < 0$



f is decreasing and concave up
$f'(x) < 0$ and $f''(x) > 0$



f is increasing and concave up
$f'(x) > 0$ and $f''(x) > 0$



Exercise: Sketch the graph of the following functions, noting:

(i) the domain,

(ii) intervals on which the function increases or decreases,

(iii) intervals on which the function is concave up or down,

(iv) local extreme points,

(v) inflection points,

(vi) horizontal and vertical asymptotes.

(a) $f(x) = x^{\frac{7}{3}} - 7x^{\frac{4}{3}}$

Domain: $(-\infty, \infty)$.

Vertical or Horizontal Asymptotes: None.

Derivatives:

$$f'(x) = \frac{7}{3}x^{\frac{4}{3}} - (7)\left(\frac{4}{3}\right)x^{\frac{1}{3}}$$

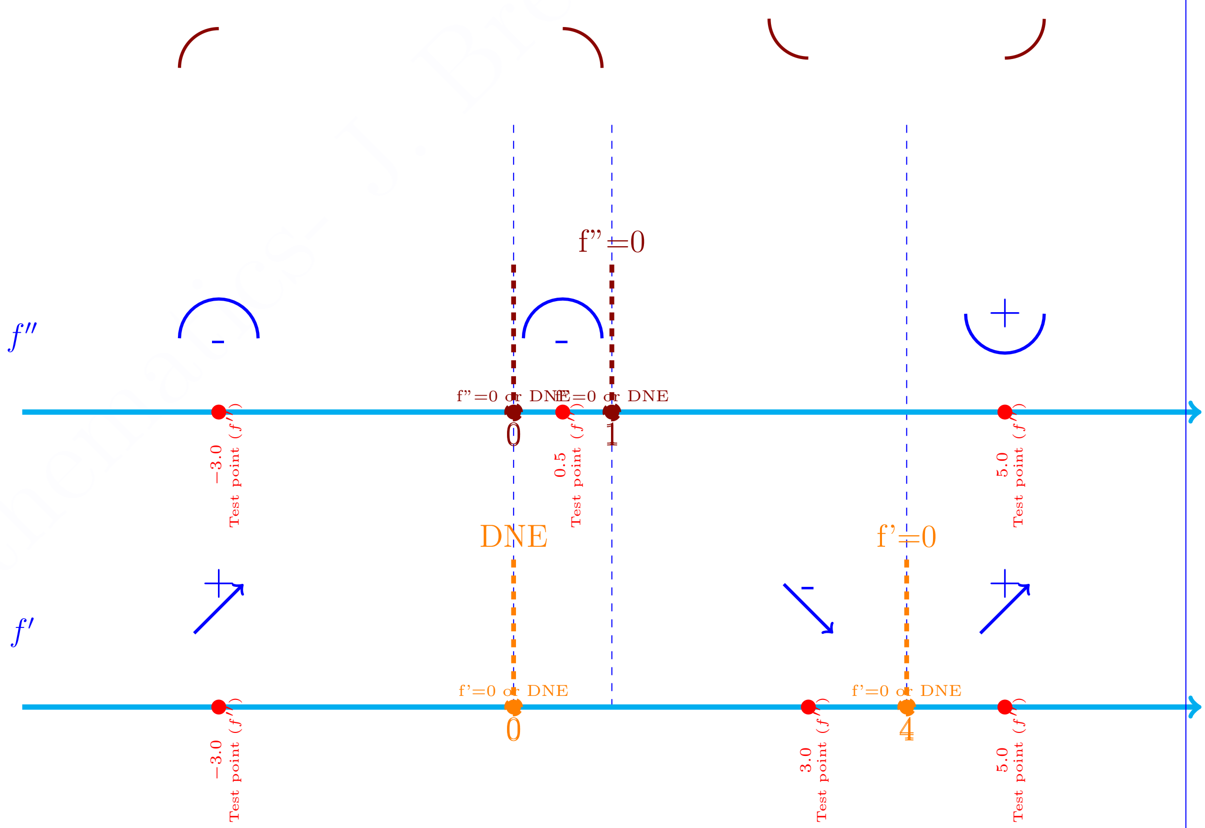
$$f'(x) = \underbrace{\frac{7}{3}x^{\frac{1}{3}}}_{\text{Factor}}(x-4).$$

$f'(x) = 0$ or f' DNE $\implies$ $x = 0$ or $x = 4$.

$$f''(x) = \frac{28}{3}x^{\frac{1}{3}} - (1)\left(\frac{28}{3}\right)x^{\frac{-2}{3}}$$

$$f''(x) = \underbrace{\frac{28}{3}x^{\frac{-2}{3}}}_{\text{Factor}}(x-1).$$

$f''(x) = 0$ or f'' DNE $\implies$ $x = 0$ or $x = 1$.



Interval of Increasing : $(-\infty, 0)$ and $(4, \infty)$

Interval of Decreasing :$(0, 4)$

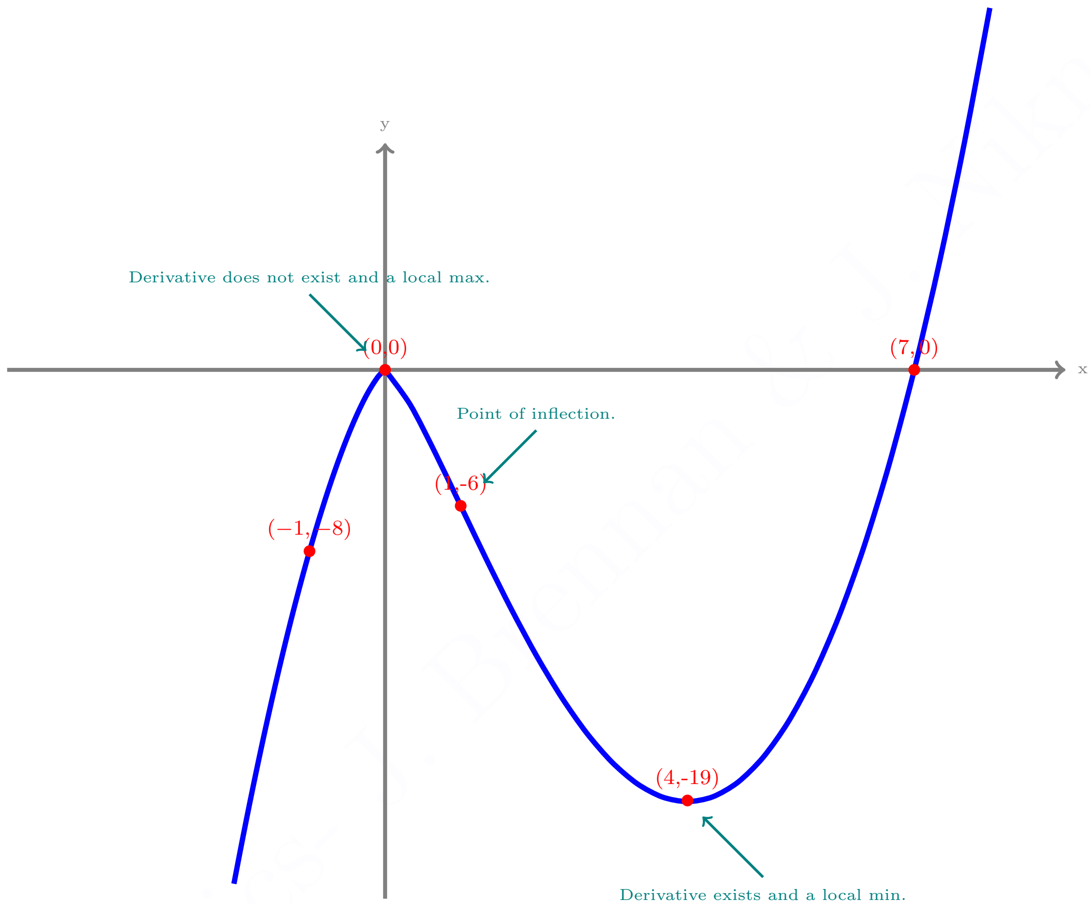
Local max: $(0, 0)$

Local Min: $(4, -\underset{\approx}{19})$

Concave up: $(1, \infty)$

Concave down: $(-\infty, 0)$ and $(0, 1)$

Inflection Point: $(1, -6)$



(b) $f(x) = \dfrac{5x}{x^2 + 3}$

Derivatives:

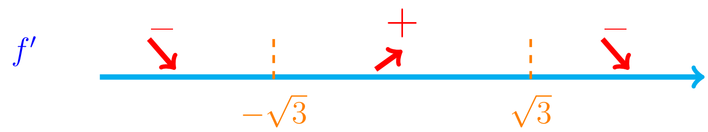
$$f'(x) = \frac{5(3 - x^2)}{(x^2 + 3)^2} \qquad f''(x) = \frac{10x(x^2 - 9)}{(x^2 + 3)^3}$$

Domain: Set $x^2 + 3 = 0$ to exclude points. No roots so Domain is $\mathbb{R}$ or $(-\infty, \infty)$

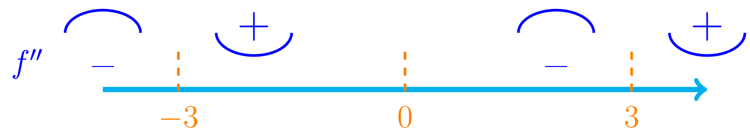
The vertical asymptotes and horizontal asymptotes of f: No vertical asymptote because of previous part. Find the limits at infinities for horizontal asymptotes. $\lim_{x \to \pm\infty} f(x) = 0 \implies \boxed{y = 0}$ is a horizontal asymptote.

i.

Intervals is f increasing? decreasing: Increasing on $\boxed{(-\sqrt{3}, \sqrt{3})}$. Decreasing on $\boxed{(-\infty, -\sqrt{3}), (\sqrt{3}, \infty)}$

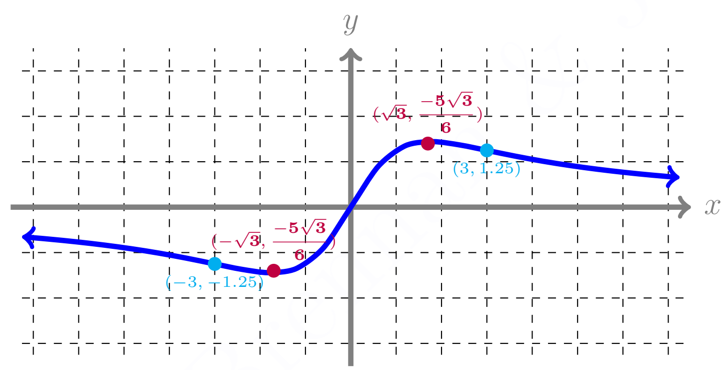


Intervals of concavity: Upward: $(-3,0),(3,\infty)$, Downward: $(-\infty,-3),(0,3)$



Local maximum or local minimum, and inflection: Local max: $(\sqrt{3},\dfrac{5\sqrt{3}}{6})$, Local Min: $(-\sqrt{3},\dfrac{-5\sqrt{3}}{6})$ Inflection points: $(-3,-1.25)$, $(3,1.25)$ and $(0,0)$

Sketch:



(c) $f(x)=\ln(x^2+2x+5)$

Domain:

Note: $x^2+2x+5=0$ has no real root,

so domain is $(-\infty,\infty)$

Asymptotes

Note: $x^2+2x+5=0$ has no real root so no vertical asymtotes.

No horizontal asymptotes because limits as $x\to\pm\infty$ is ∞.

First Derivative

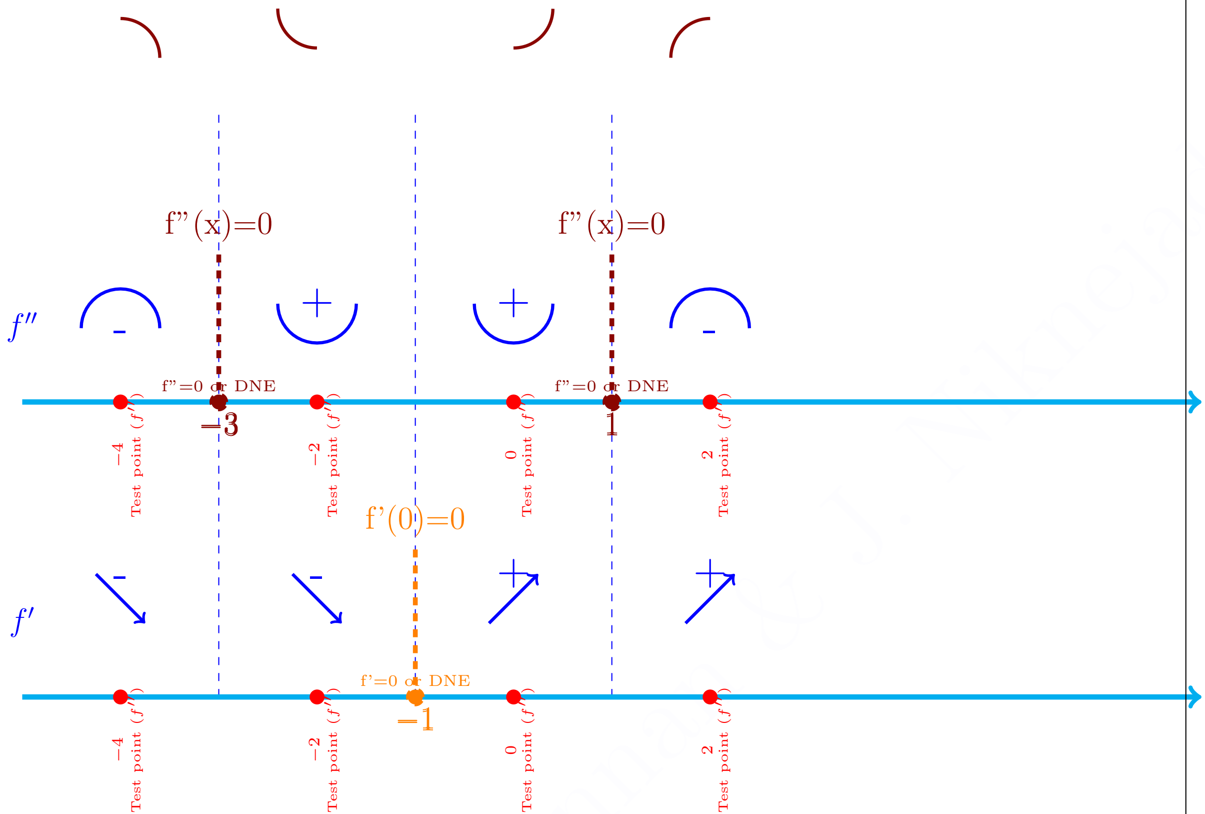
$$f'(x)=\frac{2x+2}{x^2+2x+5}$$

Second Derivative

$$f''(x)=\frac{-(2(x^2+2x-3))}{(x^2+2x+5)^2}$$

Critical Value: $\boxed{x=-1}$

Values of Interest: $\boxed{x=1}$ and $\boxed{x=-3}$



Interval of Increasing : $(-1, \infty)$
Interval of Decreasing : $(-\infty, -1)$
Local max: None
Local Min: $(-1, \ln(4))$

Concave up: $(-3, 1)$
Concave down: $(-\infty, -3)$ and $(1, \infty)$
Inflection Point: $(-3, \ln(8))$ and $(1, \ln(ln(8))$